ON THE GLOBAL ASYMPTOTIC STABILITY OF THE NAS-RIF ALGORITHM FOR BLIND IMAGE RESTORATION

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ABSTRACT

In this paper, the authors present a convergence analysis for the NAS-RIF algorithm used in blind image restoration. A novel approach is presented to determine sufficient conditions for the global convergence of the technique. The approach is general to many signal processing algorithms and incorporates Lyapunov's direct method used commonly in nonlinear system analysis.

The sufficient conditions for convergence are determined to be in the form of constraints on the blurred image pixels which can be tested for prior to the use of the NAS-RIF algorithm. An apparent trade-off between the quality of the restoration and the uniqueness of the solution is found.

1. INTRODUCTION

In many image processing applications the degradation of an image can be represented as the convolution of the true image with a blurring function also known as a point-spread function (PSF). Neglecting noise, the blurred image can be expressed by the following linear relationship [1]:

$$g(x,y) = f(x,y) * h(x,y) \qquad (1)$$

where $g(x,y)$ is the blurred image, $f(x,y)$ is the undistorted true image, and $h(x,y)$ is the PSF. The coordinates (x,y) represent the discrete pixel locations, and $*$ is the discrete linear convolution operator. In classical image restoration the true image $f(x,y)$ is recovered by using a method to invert the effect of the PSF. However, in applications such as astronomical speckle imaging and medical imaging, explicit information about the PSF is often difficult to obtain, and the image must be estimated by using *blind* image restoration methods. *Blind image restoration* is the simultaneous identification of the true image and the PSF from the blurred observation. A review of available techniques can be found in [2]. The major drawback of most existing blind image restoration algorithms is that they possess poor convergence properties: they often exhibit ill-convergence or require high computational complexity. The NAS-RIF algorithm is one of the most promising techniques for blind image restoration because it has superior convergence properties to those of its class [3]. In [3] convexity of the NAS-RIF cost function for an infinite and finite number of filter parameters has been discussed, however, uniqueness of the solution is not guaranteed. We show here the conditions under which a unique solution is guaranteed for a finite number of filter parameters.

Because of the nonlinear nature of the technique, we exploit nonlinear system analysis concepts to determine sufficient conditions for proper convergence of the scheme. In Section 2 we provide a brief review of Lyapunov's direct method for nonlinear system analysis. In Section 3 we introduce the NAS-RIF algorithm and analyze its stability properties. Sufficient conditions for the stability of the solution are determined, and an apparent trade-off between the accuracy and the uniqueness of the solution is discussed. Conclusions and final comments are presented in Section 4.

2. NONLINEAR SYSTEM ANALYSIS

2.1. Algorithm Stability

The convergence of a signal processing algorithm is related to its stability. In this section, we introduce some preliminary concepts used for our stability analysis.

We focus on discrete time systems described by a recursive time domain relationship of the form

$$\mathbf{u}_{k+1} = \mathbf{F}_k(\mathbf{u}_k) \qquad (2)$$

where $\mathbf{u}_k \in \mathcal{R}^n$, and $\mathbf{F}_k : \mathcal{R}^n \rightarrow \mathcal{R}^n$ for all $k \geq 0$. The structure of Eq. (2) guarantees that there is a unique sequence $\{\mathbf{u}_k\}$ associated with a specific initial condition $\mathbf{u}_0$. In most signal processing algorithms we try to find the parameter set $\mathbf{u}^*$ for which the algorithm "converges". This implies that we want to find a $\mathbf{u}^*$ such that it is invariant under the mapping $\mathbf{F}_k$ (i.e., $\mathbf{u}^* = \mathbf{F}_k(\mathbf{u}^*)$). Such a parameter set is called an ***equilibrium solution*** of (2).

Given an initial condition $\mathbf{u}_0$, one is often concerned with whether or not the recursion will lead to an equilibrium solution $\mathbf{u}^*$. Furthermore, one would like the recursion to converge to $\mathbf{u}^*$ given any $\mathbf{u}_0$. As we will see, this property is related to the global asymptotic stability properties of the specific algorithm.

A system of the form of Eq. (2) is asymptotically stable if for every $\epsilon > 0$ there exists a δ such that

$$\|\mathbf{u}_0 - \mathbf{u}^*\| < \delta \text{ implies } \|\mathbf{u}_k - \mathbf{u}^*\| < \epsilon \qquad (3)$$

for all $k \geq 0$, and there exists a $\eta > 0$ such that

$$\|\mathbf{u}_0 - \mathbf{u}^*\| < \eta \text{ implies } \|\mathbf{u}_k - \mathbf{u}^*\| \rightarrow 0 \qquad (4)$$

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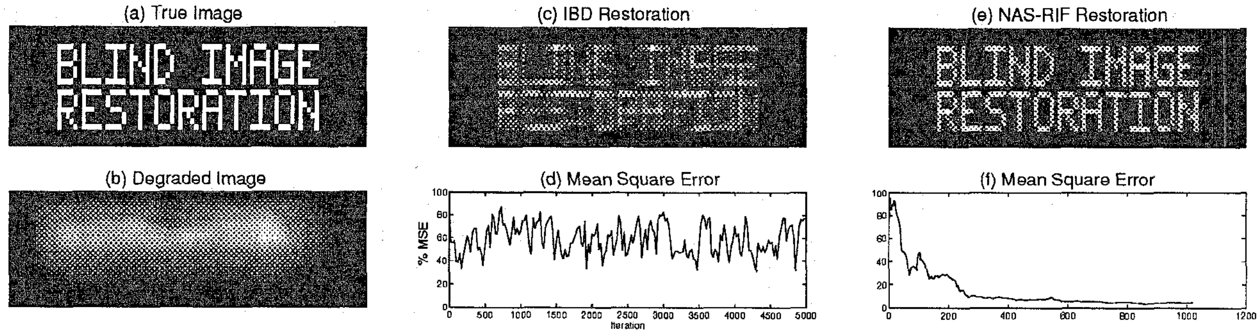

Figure 1: Simulation Results of Blind Deconvolution Algorithms to Demonstrate Stability Properties

as $k \to \infty$, where $\|\cdot\|$ is the Euclidean norm.

Condition (3) guarantees what is called *stability in the sense of Lyapunov (isL)*, and Condition (4) guarantees *attractability* of the algorithm. When both of these conditions hold for any choice of $\mathbf{u}_0$, the system is globally asymptotically stable, and the algorithm is guaranteed to converge to the equilibrium solution for any initial condition.

For reliable algorithm performance, it is not sufficient for the parameters of an algorithm to stay bounded. They must converge to the equilibrium solution. Figure 1(d) (center, bottom) shows the mean square error (MSE) of the image estimate at each iteration of the Iterative Blind Deconvolution (IBD) Algorithm [4] for a random initial image estimate. The algorithm oscillates, but does not converge to a solution. The restoration shown in Figure 1(c) (center,top) is the best estimate in the mean square sense of the IBD algorithm at the 4775th iteration. The problem is in deciding which solution to choose as the best image estimate without prior knowledge of the true image. In contrast, Figure 1(f) (right, bottom) shows the MSE of the image estimate for the NAS-RIF algorithm with an initial inverse filter setting of one in the center and zero elsewhere. The algorithm converges to a good estimate of the true image. The NAS-RIF algorithm exhibits *asymptotic stability* from this particular initial inverse filter setting because it converges to a good estimate of the true image. In this paper, we find conditions which guarantee that the NAS-RIF algorithm will converge to a good image estimate from any initial condition and for an arbitrary blurred image. That is, we want to determine the conditions to guarantee global asymptotic stability of the algorithm, which is essential for the reliability of a blind deconvolution algorithm.

2.2. Introduction to Lyapunov's Direct Method for Discrete Time Systems

Lyapunov's direct method is a powerful analysis technique which has broad application to nonlinear systems. It can be used to provide sufficient conditions for asymptotic stability of a given nonlinear system of the form of Eq. (2). A comprehensive explanation of Lyapunov theory is found in [5].

Lyapunov analysis entails the selection of an "energy" function commonly referred to as a Lyapunov function $V: \mathcal{R}^n \to \mathcal{R}$, which maps the parameter states of a given nonlinear system to a scalar quantity. If a function V can be found which exhibits certain properties, which we will explain, then the nonlinear system is globally asymptotically stable. That is, convergence of the algorithm to the equilibrium solution from any initial condition is guaranteed. We use the following theorem to find sufficient conditions for global asymptotic stability of the NAS-RIF algorithm.

Theorem 1 (Global Asymptotic Stability) *The equilibrium $\mathbf{u}^*$ of Eq. (2) is globally asymptotically stable if there is a function $V: \mathcal{R}^n \to \mathcal{R}$ such that*

(1) $V(\mathbf{u}^*) = 0$.

(2) *there are continuous, strictly increasing functions $\alpha: \mathcal{R} \to \mathcal{R}$, and $\beta: \mathcal{R} \to \mathcal{R}$ where $\alpha(0) = \beta(0) = 0$ and*

$$\alpha(\|\mathbf{u}-\mathbf{u}^*\|) \le V(\mathbf{u}) \le \beta(\|\mathbf{u}-\mathbf{u}^*\|) \qquad (5)$$

for all $\mathbf{u} \in \mathcal{R}^n$.

(3) *V is radially unbounded. i.e., $V(\mathbf{u}) \to \infty$ as $\|\mathbf{u}\| \to \infty$.*

(4) $\Delta V_k = V(\mathbf{u}_{k+1}) - V(\mathbf{u}_k) < 0$ *for all $k \ge 0$.*

3. THE NAS-RIF ALGORITHM

3.1. General Algorithm Description

The Nonnegativity and Support Constraints Recursive Inverse Filtering (NAS-RIF) technique [3] is applicable to situations in which an object of finite support is imaged against a uniformly grey background. It is comprised of a 2-D variable FIR filter $u(x,y)$ of dimension $N_{xu} \times N_{yu}$ with the blurred image pixels $g(x,y)$ as input. The output of this filter represents an estimate of the true image $\hat{f}(x,y)$. This estimate is passed through a nonlinear filter which uses a non-expansive mapping to project the estimated image into the space representing the known characteristics of the true image. The difference between this projected image $\hat{f}_{NL}(x,y)$ and $\hat{f}(x,y)$ is used as the error signal to update the variable filter $u(x,y)$. Figure 2 gives an overview of the scheme. The image is assumed to be nonnegative with known support, so the *NL* block of Figure 2 represents the projection of the estimated image onto the set of images that are nonnegative with given finite support. The cost function used in the restoration procedure is defined as:

$$J(\mathbf{u}) \;\; = \sum_{(x,y)\in D_{sup}} \hat{f}^2(x,y) \left[\frac{1-\operatorname{sgn}(\hat{f}(x,y))}{2}\right]$$

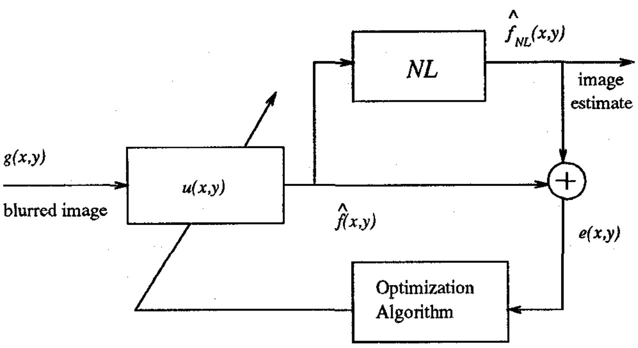

Figure 2: Proposed Blind Deconvolution Scheme for Images

$$+\sum_{(x,y)\in\overline{D}_{sup}} [\hat{f}(x,y)-L_B]^2+\gamma\left(\sum_{\forall(x,y)} u(x,y)-1\right)^2 \quad (6)$$

where $\hat{f}(x,y) = g(x,y) * u(x,y)$, and $\text{sgn}(f) = -1$ if $f < 0$ and $\text{sgn}(f) = 1$, if $f \geq 0$. D_{sup} is the set of all pixels inside the region of support, and $\overline{D}_{sup}$ is the set of all pixels outside the region of support. The variable γ in third term of Eq. (6) is nonzero only when L_B is zero, i.e., the background colour is black. The third term is used to constrain the parameters away from the trivial all-zero global minimum for this situation. A descent optimization algorithm is used for the minimization of Eq. (6). Analysis shows that the global minimum $u^*(x,y)$ of J can be made arbitrarily close to the inverse of the PSF (under ideal, noiseless conditions) by increasing the size of the dimensions of the FIR filter $u(x,y)$ [6].

3.2. Stability Analysis

A numerical descent optimization algorithm is used for the minimization of (6). A good survey of the various descent routines available is found in [7]. The algorithm can be represented by the following update law:

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \mu_k \mathbf{d}(\mathbf{u}_k) \quad (7)$$

where $\mathbf{u}_k$ is the lexicographically ordered vector of the FIR filter coefficients at the kth iteration, and μ_k is the associated step size at the kth iteration. $\mathbf{d}(\mathbf{u}_k)$ is the "direction" vector related to the particular numerical descent routine employed; it is often related to the gradient of J at $\mathbf{u}_k$, and ensures that $J(\mathbf{u}_{k+1}) \leq J(\mathbf{u}_k)$. Since, the algorithm of (7) is in the same form as that of (2), we apply Theorem 1 to evaluate its stability properties.

We choose a Lyapunov function $V(\mathbf{u})$ and determine the conditions under which the algorithm is globally asymptotically stable. Because the NAS-RIF algorithm incorporates the minimization of the cost function $J(\mathbf{u})$ a good choice for $V(\mathbf{u})$ is

$$V(\mathbf{u}) = J(\mathbf{u}) - J(\mathbf{u}^*) \quad (8)$$

where $\mathbf{u}^*$ is the desired global minimum of $J(\mathbf{u})$. The global minimum is an equilibrium solution of the descent procedure of Eq. (7) because $\mathbf{d}(\mathbf{u}^*) = \mathbf{0}$ since no modification of $\mathbf{u}^*$ can further reduce the cost.

To ensure global asymptotic stability, we must restrict $V(\mathbf{u})$, or equivalently $J(\mathbf{u}) - J(\mathbf{u}^*)$, to the four conditions outlined in Theorem 1. From this, we can obtain constraints on the blurred image pixels $g(x,y)$ to ensure proper convergence. In other words, given the NAS-RIF algorithm, we will determine a set of blurred images for which proper convergence to a good image estimate can be achieved.

The conditions given in Theorem 1 easily translates to the following constraints on $J(\mathbf{u})$:

(J1) $\alpha(\|\mathbf{u}-\mathbf{u}^*\|) + J(\mathbf{u}^*) \leq J(\mathbf{u}) \leq \beta(\|\mathbf{u}-\mathbf{u}^*\|) + J(\mathbf{u}^*)$ for all $\mathbf{u} \in \mathcal{R}^n$, where $\alpha(\cdot)$ and $\beta(\cdot)$ are as stated in (2) of Theorem 1.

(J2) $J(\mathbf{u}) \to \infty$ as $\|\mathbf{u}\| \to \infty$.

(J3) $\Delta V_k = J(\mathbf{u}_{k+1}) - J(\mathbf{u}_k) < 0$ for all $k \geq 0$ where $\mathbf{u}_k \neq \mathbf{u}^*$.

Thus, given a set of blurred images for which the associated cost function of Eq. (6) follows the three constraints listed above, convergence of the algorithm to the best image estimate for a given inverse filter size is achieved.

In this paper, we propose that the set of blurred images that satisfy the matrix condition

$$\mathbf{M} = \sum_{(x,y)\in\overline{D}_{sup}} \mathbf{g}_{xy}\mathbf{g}_{xy}^T > \mathbf{0} \quad (9)$$

where

$$\mathbf{g}_{xy} = [g(x,y)\ g(x,y-1)\cdots g(x-N_{xu}+1, y-N_{yu}+1)]^T$$

ensures global asymptotic stability of the NAS-RIF algorithm. Before we show this, we present some definitions and theorems relevant to our analysis [8].

Definition 1 (Hessian of a function) *The Hessian of a function $J : \mathcal{R}^n \to \mathcal{R}$ is defined as*

$$\nabla^2 J(\mathbf{z}) = \begin{bmatrix} \frac{\partial^2 J(\mathbf{z})}{\partial z_1^2} & \frac{\partial^2 J(\mathbf{z})}{\partial z_1 \partial z_2} & \cdots & \frac{\partial^2 J(\mathbf{z})}{\partial z_1 \partial z_n} \\ \frac{\partial^2 J(\mathbf{z})}{\partial z_2 \partial z_1} & \frac{\partial^2 J(\mathbf{z})}{\partial z_2^2} & \cdots & \frac{\partial^2 J(\mathbf{z})}{\partial z_2 \partial z_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 J(\mathbf{z})}{\partial z_n \partial z_1} & \frac{\partial^2 J(\mathbf{z})}{\partial z_n \partial z_2} & \cdots & \frac{\partial^2 J(\mathbf{z})}{\partial z_n^2} \end{bmatrix}$$

where $\mathbf{z}$ is an n-dimensional vector of comprised of components $z_1, z_2, \ldots z_n$.

Definition 2 (Convex and Strictly Convex Functions) *A function $J : \mathcal{R}^n \to \mathcal{R}$ is said to be* convex *on $\mathcal{R}^n$ when for all $\alpha \in]0,1[$ and $\mathbf{z}_1 \neq \mathbf{z}_2$ there holds*

$$J(\alpha\mathbf{z}_1 + (1-\alpha)\mathbf{z}_2) \leq \alpha J(\mathbf{z}_1) + (1-\alpha)J(\mathbf{z}_2) \quad (10)$$

It is strictly convex *if*

$$J(\alpha\mathbf{z}_1 + (1-\alpha)\mathbf{z}_2) < \alpha J(\mathbf{z}_1) + (1-\alpha)J(\mathbf{z}_2) \quad (11)$$

Theorem 2 *Let $J(\mathbf{z})$ be twice differentiable in $\mathcal{R}^n$. Then,*

1. *J is convex for all $\mathbf{z} \in \mathcal{R}^n$ if $\nabla^2 J(\mathbf{z})$ is positive semi-definite (i.e., $\nabla^2 J(\mathbf{z}) \geq \mathbf{0}$).*
2. *J is strictly convex for all $\mathbf{z} \in \mathcal{R}^n$ if $\nabla^2 J(\mathbf{z})$ is positive definite (i.e., $\nabla^2 J(\mathbf{z}) > \mathbf{0}$).*

It can be shown that the Hessian of the cost function defined in Eq. (6) is given by

$$\nabla^2 J(\mathbf{u}) = 2 \sum_{(x,y)\in D_{sup}} \mathbf{g}_{xy}\mathbf{g}_{xy}^T \left[\frac{1-\text{sgn}(\hat{f}(x,y))}{2}\right] + 2 \sum_{(x,y)\in \overline{D}_{sup}} \mathbf{g}_{xy}\mathbf{g}_{xy}^T + 2\gamma \mathbf{v}_{Ones}\mathbf{v}_{Ones}^T \quad (12)$$

where $\hat{f}(x,y) = g(x,y) * u(x,y)$, and $\mathbf{v}_{Ones}$ is a $N_{xu}N_{yu} \times 1$ column vector with all elements equal to 1.

Since $\mathbf{v}_{Ones}\mathbf{v}_{Ones}^T$ and $\mathbf{g}_{xy}\mathbf{g}_{xy}^T$ are positive semi-definite matrices, and the sum of positive semi-definite matrices is also positive semi-definite, we conclude that $\nabla^2 J(\mathbf{u})$ is positive semi-definite. (i.e., $\nabla^2 J(\mathbf{u}) \geq \mathbf{0}$). Furthermore, using the proposed matrix condition of (9), the second term of Eq. (12) is positive definite. Thus, we find that

$$\nabla^2 J(\mathbf{u}) > \mathbf{0}. \quad (13)$$

Using Theorem 2, it can be shown that $J(\mathbf{u})$ is strictly convex and is, therefore, governed by the inequality of (11). Based on this, we will now show that $J(\mathbf{u})$ follows the Conditions (J1)-(J3).

From (11) we see that $J(\mathbf{u})$ increases along any ray originating from $\mathbf{u}^*$. In addition, it can be seen from (6) that $J(\mathbf{u})$ is finite for all finite $\mathbf{u}$. Thus, there exist constants $0 < \kappa < \lambda < \infty$ such that

$$\kappa\|\mathbf{u}-\mathbf{u}^*\|^2 \leq J(\mathbf{u}) - J(\mathbf{u}^*) \leq \lambda\|\mathbf{u}-\mathbf{u}^*\|^2. \quad (14)$$

This fulfils Condition (J1). Furthermore, the increasing magnitude of $J(\mathbf{u})$ along any ray originating from $\mathbf{u}^*$ ensures that $J(\mathbf{u}) \to \infty$ as $\|u\| \to \infty$ (Condition (J2)).

Condition $(J3)$ is also fulfilled as the update law of (7) ensures that $J(\mathbf{u})$ is always decreased. Again, based on the inequality of (11), it is always possible to decrease the cost using a descent routine update law, until it reaches the equilibrium $\mathbf{u}^*$. Therefore,

$$J(\mathbf{u}_{k+1}) - J(\mathbf{u}_k) < 0$$

for all k where $\mathbf{u}_k \neq \mathbf{u}^*$.

Thus, from Theorem 1, Condition (9) ensures that the NAS-RIF algorithm is globally asymptotically stable. This condition is analogous to the requirement of *persistence of excitation* encountered in adaptive control; the blurred image must "excite" the system sufficiently, or equivalently stated, it must contain enough information to construct a good estimate of the true image. If Condition (9) does not hold, then $\nabla^2 J(\mathbf{u}) \geq \mathbf{0}$ and the solution may not be unique. This arises from Eq. (10) which implies that the cost function could be potentially flat at the global minimum. That is, there may be an infinitum of equilibrium soltions that globally minimize the cost function. Because the dimension of $\mathbf{M}$ in Eq. (9) is equal to the total number of FIR filter coefficients, a larger size filter $\mathbf{u}$ makes it more difficult to fulfil Condition (9). However, as stated in Section 3.1 (the relevant analysis can be found in [6]), this reduces the accuracy of our image estimate. Thus, as we go farther from a good image estimate, it is easier for us to fulfil the conditions that guarantee a unique solution. The matrix inequality of (9) provides a testable condition to evaluate the convergence properties of the algorithm prior to the restoration. From our Lyapunov analysis, a FIR filter of selected dimensions $N_{xu} \times N_{yu}$ can be chosen to guarantee a unique solution using the following general procedure:

1. Select an initially small size for the filter $\mathbf{u}$.
2. Build the matrix $\mathbf{M}$ given in Eq. (9).
3. Test the rank of $\mathbf{M}$. If $\mathbf{M}$ is full rank, increase the size of $\mathbf{u}$ and go to 2. Otherwise, stop and use the preceding filter size tested as the size of $\mathbf{u}$ in the NAS-RIF algorithm.

4. CONCLUSIONS

In this paper, we use a novel method of analyzing the convergence properties of iterative nonlinear signal processing algorithms. The general analysis technique makes use of Lyapunov's Direct method and is a powerful method of determining sufficient conditions for global algorithm convergence and stability. The technique has potential for the design of nonlinear algorithms with superior convergence and stability properties.

The approach was applied to determine a sufficient condition for the global convergence of the NAS-RIF algorithm used in blind image restoration. This condition can be tested for prior to the use of the NAS-RIF algorithm to determine an optimal recursive filter size to guarantee a unique image estimate.

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